Journal of Statistical Physics, Vol. 82, Nos. 3/4, 1996

Diffusive Escape in a Nonlinear Shear Flow: Life and Death at the Edge of a Windy Cliff

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Received May 11, 1995; final July 14, 1995

The survival probability of a particle diffusing in the two-dimensional domain $x > 0$ near a "windy cliff" at $x = 0$ is investigated. The particle dies upon reaching the edge of the cliff. In addition to diffusion, the particle is influenced by a steady "wind shear" with velocity $\mathbf{v}(x, y) = v \operatorname{sign}(y)\hat{\mathbf{x}}$, i.e., no average bias either toward or away from the cliff. For this semi-infinite system, the particle survival probability decays with time as $t^{-1/4}$, compared to $t^{-1/2}$ in the absence of wind. Scaling descriptions are developed to elucidate this behavior, as well as the survival probability within a semi-infinite strip of finite width $|y| < w$ with particle absorption at $x = 0$. The behavior in the strip geometry can be described in terms of Taylor diffusion, an approach which accounts for the crossover to the $t^{-1/4}$ decay when the width of the strip diverges. Supporting numerical simulations of our analytical results are presented.

KEY WORDS: Survival probability; wind shear; Taylor diffusion.

1. INTRODUCTION

Consider a particle which diffuses in the semi-infinite planar domain $(x > 0, y)$ and is absorbed when $x = 0$ is reached. The line $x = 0$ can be viewed as a "cliff" and absorption at $x = 0$ corresponds to the particle falling to its death. For this system, it is well known that the particle survival probability $S(t)$ decays in time as (see, e.g., ref. 1)

$$S(t) \sim \frac{x_0}{(Dt)^{1/2}} \tag{1}$$

Here x_0 is the initial distance from the particle to the cliff and D is the diffuxion coefficient. In this article, we are interested in understanding the

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0022-4715/96/0200-0999$09.50/0

time dependence of $S(t)$ when the diffusing particle also experiences a "wind shear," defined as the velocity field $\mathbf{v}(x, y) = v\hat{\mathbf{x}}$ for $y > 0$ and $\mathbf{v}(x, y) = -v\hat{\mathbf{x}}$ for $y < 0$ (Fig. 1). Our primary result is that, although there is no average bias either toward or away from the cliff, the survival probability decays as $t^{-1/4}$, compared to the $t^{-1/2}$ decay in the absence of the bias. This result is contrary to the naive intuition of a faster decay, as wind shear enhances longitudinal (x) diffusion, which, from Eq. (1), should reduce the survival probability. More generally, our interest is in understanding the interplay between macroscopically heterogeneous convection and diffusion on first-passage phenomena. The wind shear geometry is a relatively simple example of such a system. In spite of this simplicity, relatively unusual first-passage characteristics occur.

Our work is also complementary to a recent paper by Lee and Koplik (2) where the survival probability of a particle in the same wind shear was

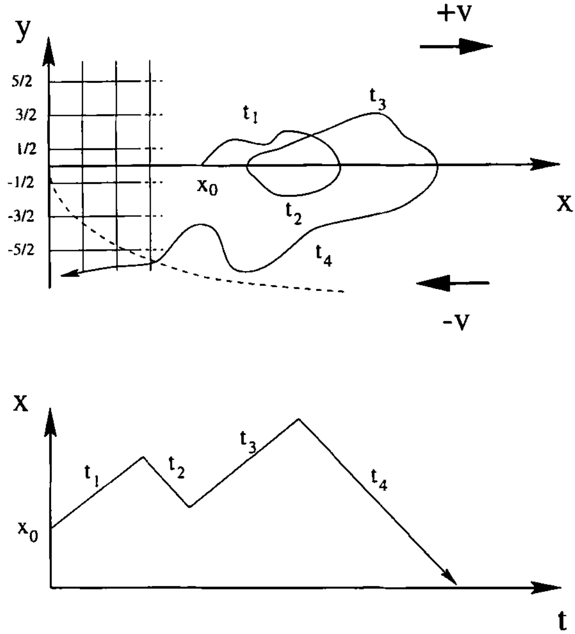

Fig. 1. Wind shear in two dimensions with a "cliff" at $x = 0$. The underlying lattice (thin lines) is defined with $y = 0$ bisecting two horizontal rows of points. A typical particle trajectory in this system is sketched. If the particle strays below the curve $y \cong -(Dx/v)^{1/2}$ (dashed), it is almost surely absorbed. The lower portion schematically shows the same trajectory plotted with x as a function of t. The distribution of segment lengths t_i in this representation is proportional to $t^{-3/2}$ (see text).

considered, but in a "two-layer" system with y unbounded, and with absorbing boundaries at $x=0$ and at $x=L$. For this system, Lee and Koplik gave asymptotic arguments to show that the survival probability decays as $\exp(-t/T)$ with $T \propto L$. In comparison, for pure diffusion in the same domain, $S(t) \sim \exp(-t/T_D)$, but with a decay time $T_D \propto L^2$. Thus, in accord with intuition, wind shear decreases the survival probability in the two-layer system [although the functional form of $S(t)$ is unchanged]. However, for the semi-infinite planar system, the wind shear has the opposite effect of enhancing the particle survival probability and moreover changes the decay exponent from 1/2 to 1/4.

A fundamental aspect of diffusive motion in wind shear is that the particle spends relatively long periods of time exclusively in the region $y>0$ or $y<0$ before returning to $y=0$. In fact, the distribution of time intervals between successive crossings of $y=0$ coincides with the first passage probability for a one-dimensional random walk to return to its starting point. Since this probability decays as $t^{-3/2}$, the particle has relatively long alternating flights where x increases or decreases linearly with time. For a particle which starts at the origin in an unbounded two-dimensional system, this structure for the longitudinal steps leads to the following unusual probability distribution in x at time t[3,4],

$$\mathscr{P}(x, t) \equiv \int dy\, P(x, y, t) \propto \frac{1}{[(vt)^2 - x^2]^{1/2}} \tag{2}$$

Rather surprisingly, the probability distribution is peaked as $x \to \pm vt$ and is a minimum for $x=0$. These unusual features are one manifestation of the classical arcsine law for long leads in one-dimensional random walk.[3] The interplay between these long excursions and the absorbing boundary is responsible for many of the intriguing features of the particle survival probability.

In Section 2 we define microscopic lattice rules to model a particle moving in a wind shear. Somewhat different behavior occurs if the underlying diffusion is only in the y (transverse) direction compared to the underlying diffusion being isotropic. While the former case is in some sense simpler, it does not have a smooth limit as bias vanishes. To understand this basic limit, we therefore introduce a variant of the original model in which the underlying diffusion is spatially isotropic. This allows for a continuum description of the process. The model with isotropic diffusion also provides useful insights in the limit as the bias vanishes. In Section 3 we present scaling arguments to determine the long-time behavior of $S(t)$ for the semi-infinite planar system for the two basic cases of anisotropic and isotropic underlying diffusion. In Section 4 we apply the Taylor diffusion description to determine the survival probability in a finite-width strip $x > 0$ and $|y| \leqslant w$, with reflection at $|y| = w$ and absorption at $x = 0$. Our results for the strip are combined with crossover arguments to provide additional insights into the corresponding first-passage properties of the semi-infinite planar system. A salient feature of the survival probability in the strip geometry is the nontrivial dependence of the survival probability on the initial distance from the particle to the cliff, the velocity, and the diffusion coefficient. While the approaches presented in this paper are nonrigorous, our results appear to be asymptotically correct. However, it would be desirable to develop more rigorous approaches to account for the behavior of the survival probability. We conclude and discuss some open questions in Section 5.

2. WIND SHEAR IN TWO DIMENSIONS

Our system is a square lattice with $x \geqslant 0$ and $y = 0$ defined to bisect the bonds which join the bottom row in the upper half-plane to the top row in the lower half-plane (Fig. 1). With this construction, there are no sites with $y = 0$, and ambiguities associated with assigning hopping rates from these sites are avoided. The hopping rates from any site in the upper (lower) half-plane are spatially homogeneous. We consider two different hopping rules which correspond, respectively, to anisotropy and isotropy (for $v \to 0$) in the underlying diffusive motion.

For anisotropic diffusion, to account for a variable transverse diffusivity and longitudinal velocity, we define the following hopping rates to six neighbors of a given site. For sites in the upper-half-plane (Fig. 2a)

$$\begin{aligned} p_{1,\pm 1} &= \tfrac{1}{4}(1+v)\, D_y \\ p_{1,0} &= \tfrac{1}{2}(1+v)(1-D_y) \\ p_{-1,\pm 1} &= \tfrac{1}{4}(1-v)\, D_y \\ p_{-1,0} &= \tfrac{1}{2}(1-v)(1-D_y) \end{aligned} \qquad \text{(anisotropic diffusion)} \qquad (3a)$$

Here the subscript on $p_{\mathbf{r}}$ indicates vector displacement defined by the hop. In the lower half-plane, similar hopping rates exist, except with an opposite sign for v. With these rules, there is a diffusivity in the x direction whose magnitude depends on v and vanishes as $v \to 1$. For the case of unit velocity and unit transverse diffusivity, these hopping rules reduce to a two-site neighborhood. This appears to be the simplest implementation for random walk modeling of diffusion in a wind shear.

While the above hopping rule is suitable for most of our purposes, it leads to pathological behavior for $v \to 0$ or for $D_y \to 0$. Because useful

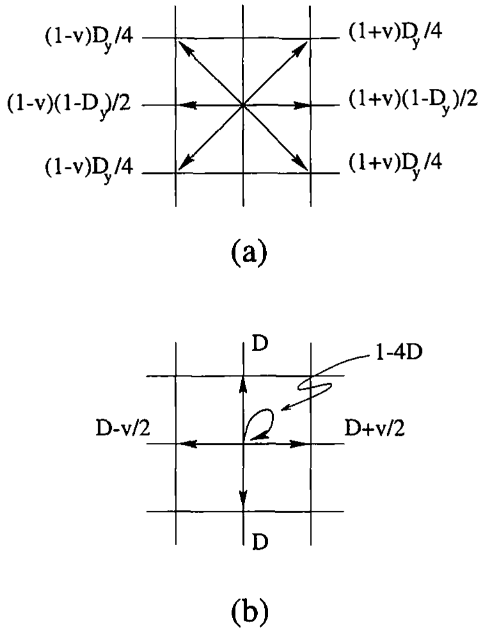

Fig. 2. Ilustration of the microscopic rules for a single hopping event for (a) anisotropic and (b) isotropic diffusion.

insights can be gained by considering the crossover to purely diffusive behavior as $v \to 0$, we therefore define a second model in which the underlying diffusion is isotropic. For sites in the upper half-plane, we define the hopping rates (Fig. 2b)

$$\begin{aligned} p_{0,\pm 1} &= D \\ p_{1,0} &= D + \frac{v}{2} \\ p_{-1,0} &= D - \frac{v}{2} \\ p_{0,0} &= 1 - 4D \end{aligned} \qquad \text{(isotropic diffusion)} \qquad (3b)$$

Clearly, these rules can be applied only for $D < 1/4$ and $v < 1/2$ to ensure positivity of all the hopping rates.

In the continuum limit, the probability distribution in the anisotropic hopping model obeys the Fokker–Planck equation,

$$\frac{\partial c(x, y, t)}{\partial t} + v\,\mathrm{sign}(y)\,\frac{\partial c(x, y, t)}{\partial x} = D_y\,\frac{\partial^2 c(x, y, t)}{\partial y^2} \tag{4}$$

Here $c(x, y, t)$ is the probability distribution at (x, y) at time t. The form of the convective term accounts for a bias along $+x$ for $y > 0$ and a bias along $-x$ for $y < 0$. For this anisotropic system, the appropriate boundary condition is somewhat counterintuitive. For $y > 0$, $c(x = 0, y, t) = 0$, corresponding to no particles being introduced into the system. For $y < 0$, once $x = 0$ is reached, a particle cannot return to the domain $x > 0$ and therefore there is no boundary condition for the half-range $(x = 0, y < 0)$. The continuum limit of the isotropic hopping model is more convenient, primarly because of a much simpler boundary condition. The corresponding Fokker–Planck equation for the probability distribution is

$$\frac{\partial c(x, y, t)}{\partial t} + v\,\mathrm{sign}(y)\,\frac{\partial c(x, y, t)}{\partial x} = D\nabla^2 c(x, y, t) \tag{5}$$

and the appropriate boundary condition is $c(x = 0, y, t) = 0$. We are typically interested in the situation where a particle starts at $y = 0$ some distance from the cliff, corresponding to the initial condition $c(x, y, t = 0) = \delta(x - x_0)\,\delta(y)$, although it is also of interest to consider starting positions not at $y = 0$.

Although Eqs. (4) and (5) are linear, the boundary value problem is not elementary. Because we have been unable to solve this problem, we resort to scaling arguments in the next section to determine the asymptotic behavior of the survival probability with the above two realizations of wind shear.

3. SURVIVAL PROBABILITY IN THE PLANAR SYSTEM

3.1. Wind Shear with Anisotropic Diffusion

Consider now the survival probability of a particle initially at $(x_0, 0)$ in the semi-infinite planar system $x > 0$, with absorption at $x = 0$. From numerical exact enumeration of the probability distribution for the case $v = D_y = 1$, it is evident that the survival probability decays as $t^{-1/4}$ (Fig. 3). A relatively simple way to understand this result is to focus on those points where the particle trajectory crosses from $y > 0$ to $y < 0$ or vice versa. Since the transverse motion is a one-dimensional random walk, the probability distribution of times between successive crossings asymptotically varies as $t^{-3/2}$.[1] Consequently, the longitudinal displacement $x(t)$

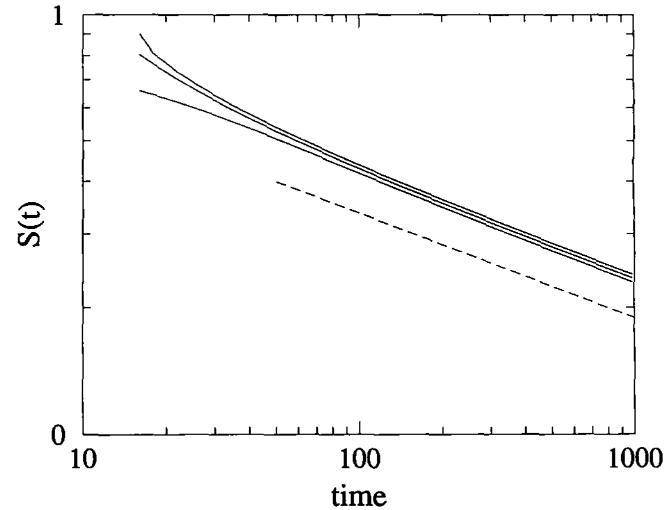

Fig. 3. Time dependence of $S(t)$, based on exact enumeration of the spatial probability distribution for two "semi"-particles (each with weight 1/2) initially at $(x_0, y_0) = (16, \pm 1/2)$, in a planar semi-infinite system with $v = 1$ and anisotropic diffusion with $D_y = 1$, 1/4, and 1/16 (upper to lower data sets). The dashed straight line has slope $-1/4$.

versus t is a Lévy flight which consists of "segments" t_i whose lengths are distributed according to the above distribution (Fig. 1).[2] Here, the term "segment" refers to a connected portion of the trajectory with the y coordinate having the same sign.

The existence of a finite observation time t, however, implies that the segment length distribution is necessarly cut off at this time. With this cutoff, the average segment length is given by $\langle t \rangle \propto \int^t t' \times t'^{-3/2} \, dt' \propto t^{1/2}$. Thus, as might be anticipated, a trajectory of t steps can typically be decomposed into $\mathcal{N} \propto \sqrt{t}$ segments, each of length $\sqrt{t}$. At the segment level, the probability that the walk does not reach $x = 0$ is equivalent to particle survival. Since there are $\mathcal{N}$ independent segments, this no-return probability should therefore vary as $1/\sqrt{\mathcal{N}}$, which, in turn, is proportional to $t^{-1/4}$, in accord with our observations.

It is also instructive to consider the full dependence of $S(t)$ on microscopic parameters. For our system, $S(t)$ can only be a function of the basic variables x_0, v, D_y, and t. Since $S(t)$ is dimensionless, it is convenient

[2] In the context of Levy flights, the survival probability has been considered using different approaches in ref. 5.

to introduce the following groupings with units of time $\tau_{\|} = x_0/v$ and $\tau_D = x_0^2/D_y$, and write for the survival probability

$$S(t) \sim f\left(\frac{t}{\tau_{\|}}, \frac{t}{\tau_D}\right) \tag{6}$$

A crucial fact is that the particle survival probability is governed by the *difference* in residence times within the regions $y > 0$ and $y < 0$. As written in the original arcsine law [Eq. (2)], this difference is *independent* of D_y. Combining this with the fact that $S \propto t^{-1/4}$, we find that the asymptotic form of $S(t)$ reduces to

$$S(t) \propto \left(\frac{\tau_{\|}}{t}\right)^{1/4} = \left(\frac{x_0}{vt}\right)^{1/4} \tag{7}$$

To justify this rather unexpected behavior, it is instructive to consider the survival probability on a finite-width strip (Section 4). For a system with finite extent, $|y| \leqslant w$, w and D_y naturally appear in the combination $\tau = w^2/D_y$. Since any finite value of D_y leads to the same value of $\tau_\perp$ as $w \to \infty$, it suggests that Eq. (7) should be independent of D_y. Our numerical data support this conclusion (Fig. 3), as long as $\tau_{\|} > \tau_D$, so that there is mixing between the upper and lower half-planes, before significant

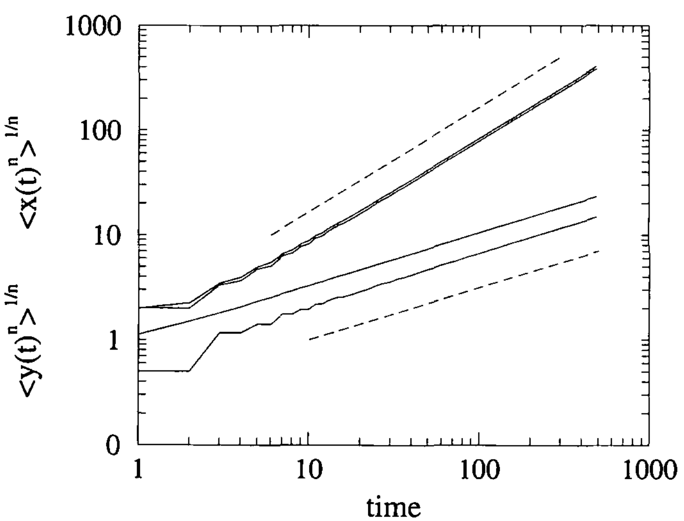

Fig. 4. Time dependence of $\langle x(t)^n \rangle^{1/n}$ (upper curves) and $\langle y(t)^n \rangle^{1/n}$ (lower curves) for $n = 1$, 2 for the distribution of surviving walks for the case $v = D_y = 1$ and $x_0 = 1$. The straight lines have slopes of 1 and 1/2.

absorption occurs. However, our data typically exhibit a very small and unexplained residual systematic dependence of $S(t)$ on D_y.

Another interesting aspect of the surviving particles is their spatial distribution (Fig. 4). Numerically, the surviving particles are predominantly within a region whose mean longitudinal and transverse positions are given by $\langle x(t)\rangle \propto t$ and $\langle y(t)\rangle \propto t^{1/2}$. These results can be justified heuristically. For walks to survive, there clearly must be a longer residence time in the upper half-plane than the lower half-plane. However, starting from $(x, 0)$, a walk can tolerate a short excursion into the lower half-plane and still survive. If this excursion extends to a transverse distance $y \cong -(Dx/v)^{1/2}$, then the time required for the particle to return to "safety" $(y > 0)$ becomes of the order of the time for the particle to be convected to the cliff. Thus, if the particle enters the region $y < -(Dx/v)^{1/2}$, it is likely to be absorbed (Fig. 1). Since surviving particles must almost always be in the region $y > -(Dx/v)^{1/2}$ and hence predominantly in the upper half-plane, this leads to the longitudinal displacement of the survivors being proportional to t. In a similar spirit, if the survivors are mostly in the upper half-plane, then their transverse dispersion can be governed only by diffusion, so that $\langle y(t)\rangle \propto t^{1/2}$.

It is also worth noting that if $\langle x(t)\rangle \propto t$, then the typical value of y for which absorption occurs is $-(Dt/v)^{1/2}$. Thus the two-dimensional system can be reduced to an effective one-dimensional transverse problem of a purely diffusing particle starting in the domain $y > 0$ with an absorbing boundary at $-(Dt/v)^{1/2}$. This latter system has been extensively studied.[3] It is known that $S(t) \propto t^{-\alpha}$ with α dependent on D and v in such a way that $\alpha \to 1/2$ for $v \to 0$ and $\alpha \to 0$ for $v \to \infty$. This connection between the wind shear and one-dimensional moving boundary problems provides further evidence of a decay exponent for $S(t)$ in wind shear which is less that 1/2.

3.2. Wind Shear with Isotropic Diffusion

When the underlying diffusion is isotropic, the survival probability again decays as $t^{-1/4}$ in the long-time limit (Fig. 5a). As might be anticipated, the effect of diffusion is subdominant with respect to convection in governing the value of the exponent in the time dependence of $S(t)$. However, for the isotropic system there is a crossover from diffusive behavior for small v to convective behavior as v becomes large. By applying scaling to determine the nature of this crossover, we also determine the dependence of $S(t)$ on the physical parameters of the system. This aspect

[3] See, e.g., ref. 6. A pedagogical account is given in ref. 7.

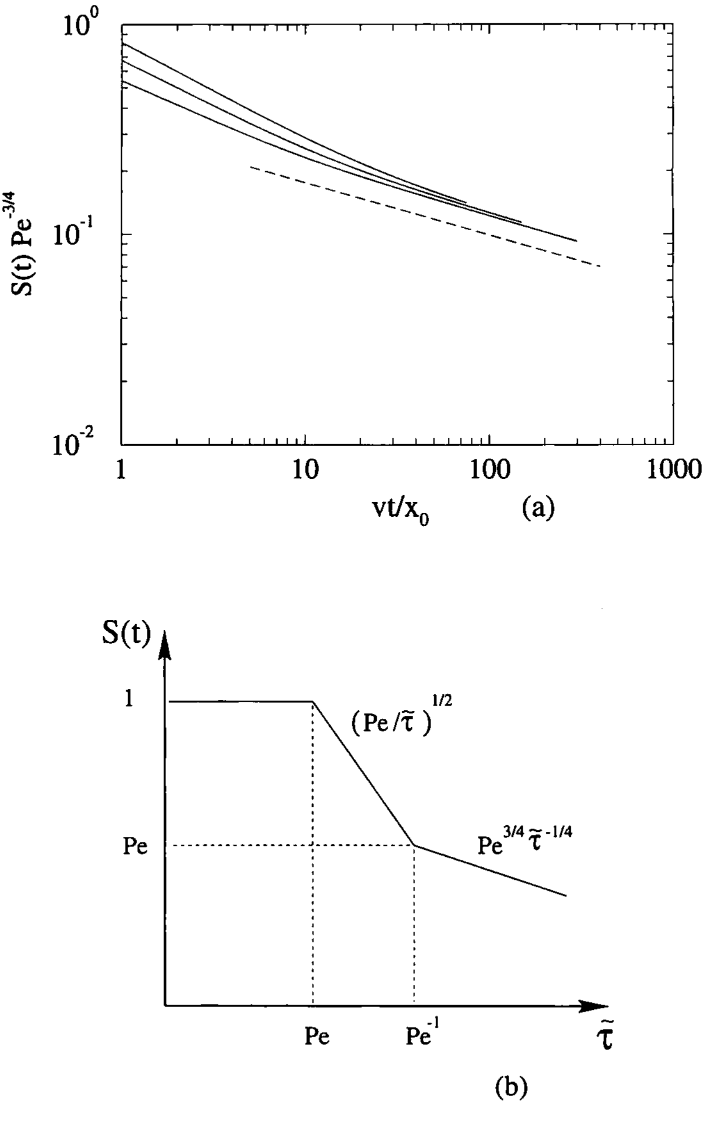

Fig. 5. (a) Representative results for the time dependence of the $S(t)$ based on exact enumeration of the probability distribution for a planar semi-infinite system with isotropic diffusion and (i) $(v, x_0, D) = (0.2, 1, 0.25)$, or $Pe = 0.8$, (ii) $(0.1, 1, 0.25)$, $Pe = 0.4$, and (iii) $(0.05, 1, 0.25)$ or $Pe = 0.2$ (lower to upper data sets, respectively). Plotted is $S(t)\, Pe^{-3/4}$ versus $\tilde{\tau} = vt/x_0$. The dashed line has slope $-1/4$. (b) Schematic behavior for $S(t)$ versus t on a double logarithmic scale with isotropic diffusion in the limit of $Pe \gg 1$.

merits emphasis, as there are important differences in the dependence of $S(t)$ on system parameters with isotropic and anisotropic diffusion.

For isotropic diffusion, the basic time scales of the system are

$$\tau_D = \frac{x_0^2}{D}, \qquad \tau_{\parallel} = \frac{x_0}{v}, \qquad \tau_{\times} = D/v^2 \tag{8a}$$

These times are, respectively, the characteristic time to diffuse to the cliff, the time to convect to the cliff, and the crossover time beyond which convection dominates over diffusion. It is convenient to measure these times in units of $\tau_{\parallel}$ to give

$$\tilde{\tau}_D = Pe, \qquad \tilde{\tau}_{\parallel} = 1, \qquad \tilde{\tau}_{\times} = \frac{1}{Pe} \tag{8b}$$

where $Pe = x_0 v/D$ is the Péclet number. In the case where $Pe > 1$, convection becomes established before the particle can reach the cliff and the behavior is the same as that discussed previously for anisotropic diffusion. On the other hand, for $Pe < 1$, the fundamental times obey the inequalities $\tilde{\tau}_D < \tilde{\tau}_{\parallel} < \tilde{\tau}_x$. Consequently, particle death at early times is determined by diffusion and it is only for $t > \tilde{\tau}_x$ that convective behavior sets in (Fig. 5b). In dimensionless units, the early-time decay has the form $S(t) \sim (Pe/\tilde{\tau})^{1/2}$, with $\tilde{\tau} = vt/x_0$, while at late times the survival probability can be written as $S(\tau \to \infty) \sim A(Pe)\, \tilde{\tau}^{-1/4}$. Matching these two asymptotes at $\tilde{\tau}_x = 1/Pe$ fixes the amplitude $A(Pe)$, from which

$$S(t \to \infty) \propto \frac{(Pe)^{3/4}}{\tilde{\tau}^{1/4}} = \frac{x_0 v^{1/2}}{D^{3/4} t^{1/4}} \tag{9}$$

As shown in Fig. 5a, the long-data accord with scaling behavior predicted by Eq. (9).

4. SURVIVAL IN THE SEMI-INFINITE STRIP

It is instructive to examine the survival probability in the presence of wind shear in a semi-infinite strip of width $|y| \leqslant w$. Because this system is effectively one dimensional, the longitudinal motion asymptotically reduces to diffusion in one dimension. However, this motion is properly described by Taylor diffusion, which accounts for the interplay between diffusion and convection.[8] From this description, the form of the survival probability is, in principle, straightforward to deduce. It is then possible to infer properties of the survival probability in the semi-infinite planar system by letting $w \to \infty$. However, there is an unexpected subtlety in the behavior of $S(t)$

for the strip, which depends on the relation between x_0 and w. The resolution of this feature provides useful general insights about the nature of the survival probability.

Taylor diffusion arises because the particle convects to the right while $y>0$ and convects to the left when $y<0$. Since this switching between $y>0$ and $y<0$ is governed by diffusion, the longitudinal motion is also diffusive. The characteristic time to switch between right-going and left-going segments is the same as the time needed to diffuse between the regions $y>0$ and $y<0$ (or *vice versa*), namely, $\tau_\perp \sim w^2/D$. Therefore the step length of the effective longitudinal random walk is $l \sim v\tau_\perp$ and the corresponding Taylor diffusivity is given by $D_\| \sim l^2/\tau_\perp \sim v^2w^2/D$.[8]

With this characterization of the longitudinal motion, we now investigate the survival probability in the strip in terms of the two basic time scales $\tau_\|$ and $\tau_\perp$. First consider the limit $\tau_\| \gg \tau_\perp$, which can be reexpressed as $x_0 \gg l$. Physically, many longitudinal segments are needed before the cliff is reached, or equivalently, the walk encounters the sides of the strip many times before reaching $x=0$. Therefore, before any absorption occurs, the trajectory has time to become truly one dimensional. Thus we conclude that the particle survival probability is given by the suitably adapted one-dimensional expression,

$$S(t; x_0 \gg l) \sim \frac{x_0}{(D_\| t)^{1/2}} \tag{10a}$$

In terms of the basic time scales, this (dimensionless) survival probability can be reexpressed as

$$S(t; x_0 \gg l) \sim \frac{\tau_\|}{(\tau_\perp t)^{1/2}} \tag{10b}$$

However, the converse limit $\tau_\| \ll \tau_\perp$ $(x_0 \ll l)$ is more relevant for understanding the survival probability in the planar system with wind shear. In this case, there is significant loss of probability by absorption at $x=0$ before the sides of the strip play any role. Consequently, the survival probability should decay as $(x_0/vt)^{1/4}$ at short times, as in the semi-infinite planar system, and cross over to a decay of the form $Bt^{-1/2}$ when $t \ll \tau_\perp$ (Fig. 6). By matching these two asymptotes at $t=\tau$ we determine the amplitude B and thereby find for the survival probability

$$S(t; x_0 \ll l) \propto \left(\frac{\tau_\| \tau_\perp}{t^2}\right)^{1/4} \tag{11}$$

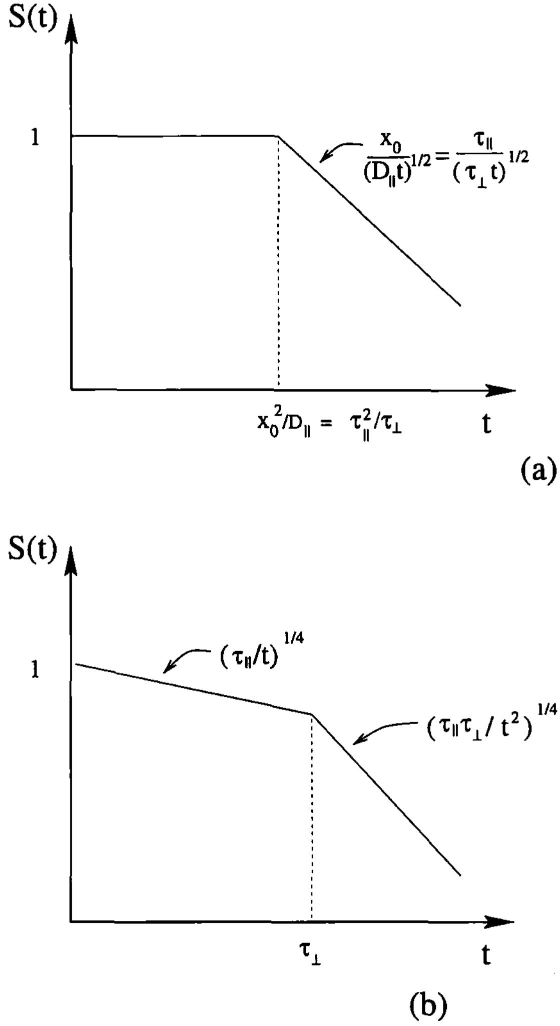

Fig. 6. Schematic behavior for $S(t)$ versus t on a double logarithmic scale for a finite strip of width w with anisotropic diffusion in the limits of (a) $x_0 \gg l$ and (b) $x_0 \ll l$.

A noteworthy feature of (10) and (11) is the opposite dependences in w. While $S(t; x_0 \gg l) \propto 1/w$, $S(t; x_0 \ll l) \propto w^{1/2}$ (and is also proportional to $x_0^{1/4}$). In the latter case, $S(t; x_0 \ll l)$ must be increasing in w for there to be a slower time dependence in $S(t; x_0 \ll l)$ when $w = \infty$. The changeover from (10) to (11) as x_0 becomes smaller than l can be viewed as arising by x_0 "sticking" at l. This occurs because after one Taylor diffusion step, the walk is either absorbed or is reintroduced at a distance of order l from the absorber. In the time $\tau_\perp \sim w^2/D$ required for this reintroduction, only a fraction $(x_0/v\tau_\perp)^{1/4}$ of the walks remain. Consequently,

$$\begin{aligned} S(t; x_0 \ll l) &\cong S(t; x_0 \gg l)|_{x_0=l} \times \left(\frac{x_0}{v\tau_\perp}\right)^{1/4} \\ &\propto \frac{x_0^{1/4} w^{1/2}}{(Dv)^{1/4}} \end{aligned} \tag{12}$$

Notice also that the limiting expressions for $S(t)$ for $x_0 \gg l$ and $x_0 \ll l$ both become of order $w/(Dt)^{1/2}$ when $x_0 \cong l$. This provides a useful self-consistency check on our approach.

5. DISCUSSION

We have investigated the behavior of the survival probability for a diffusing particle with a planar absorbing boundary, or cliff, in the presence of a superimposed "wind shear." Although this velocity field does not have any bias either toward or away from the cliff, the long-time behavior of the survival probability $S(t)$ is strongly affected by the wind shear. Our primary result is that $S(t) \propto t^{-1/4}$ for a semi-infinite plane, compared to a decay of $t^{-1/2}$ for the survival probability in the absence of bias. Although our approaches are neither rigorous nor microscopic, they provide a good quantitative account of simulation results. It is worth noting that the image method, which yields the survival probability in the presence of a planar absorber for both diffusion and uniform convection, does not appear to be generalizable to wind shear.

Our prediction for the survival probability has applicability beyond the case of wind shear. Consider, for example, a particle diffusing with superimposed linear shear, $\mathbf{v}(x, y) = vy\mathbf{x}$, and with absorption at $x = 0$. For this system, exact enumeration of the probability distribution in a lattice version of the system shows that the survival probability decays as $t^{-1/4}$.[9] This result is also supported by the heuristic segment argument given in Section 3.1, which suggests that $S(t) \propto (\tau_{||}/t)^{1/4}$, where $\tau_{||}$, now the typical time to convect from x_0 to the cliff in shear flow, is determined by $v(D\tau_{||})^{1/2}\,\tau_{||} = x_0$. Thus for the linear shear, our scaling approach predicts the following asymptotic dependence of the survival probability on model parameters:

$$S(t) \propto \left(\frac{x_0 D}{v}\right)^{1/6} (Dt)^{-1/4} \tag{13}$$

In fact, we believe that the $t^{-1/4}$ decay should hold for any velocity field with $v_x(y) = -v_x(-y)$. For example, for power-law shear $\mathbf{v}(x, y) = vy\,|y|^{\beta-1}\hat{\mathbf{x}}$, the asymptotic form

$$S(t) \propto \left(\frac{x_0 D}{v}\right)^{1/[2(\beta+2)]} (Dt)^{-1/4} \tag{14}$$

is expected. Notice that only in the case of wind shear ($\beta = 0$) is $S(t)$ independent of D. Another situation for which unusual behavior of $S(t)$ can be anticipated is stratified random flow[4] in which $v_x(y)$ is a random zero-mean function of y. In this case, there could be different behavior for $S(t)$ in a typical configuration of the velocity field and when averaged over all configurations of velocities.

It would also be desirable to develop more rigorous and microscopic approaches to understand the first-passage characteristics of systems with various types of neutral bias fields. For the case of wind shear, the distinct nature of the problem for $y > 0$ and $y < 0$ suggests that the Wiener–Hopf technique may be suitable. This technique has been successfully applied to a related problem involving first passage in the presence of colored noise.[12] However, we have been unable to apply this method to our problem.

ACKNOWLEDGMENTS

We thank A. Acrivos, D. ben-Avraham, E. Ben-Naim, C. R. Doering, A. Drory, J. Koplik, and J. Lee for helpful discussions and E. Ben-Naim for providing the simulation results for the survival probability in linear shear flow. We also gratefully acknowledge NSF grant DMR-9219845 for partial support of this research.

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